14.3 29) Suppose $f(x,y)$ is continuous over a region R in the plane and that the area $A(R)$ is defined. If there are constants m and M such that $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, prove that

$$mA(R) \leq \iint_R f(x,y)\,dA \leq MA(R).$$

Sol'n: Since f is continuous over R, it is integrable on R (by Prop 2 of lecture). We have

$$\iint_R f(x,y)\,dA \underset{\text{assumption}}{\leq} \iint_R M\,dA \underset{\text{Prop 3.}}{=} M\iint_R 1\,dA \underset{\text{Def 3.}}{=} MA(R)$$

Similarly,

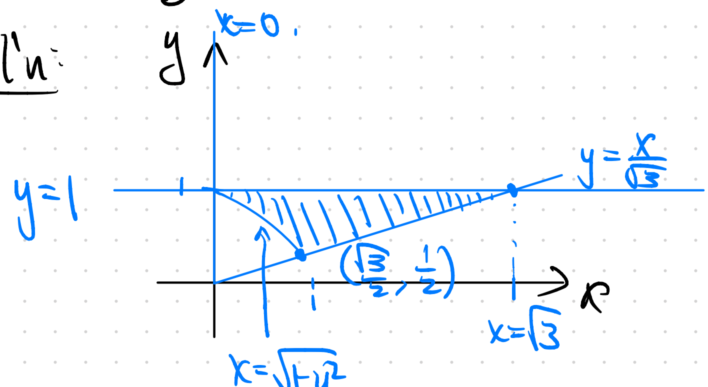
$$\iint_R f(x,y)\,dA \underset{\text{assumption}}{\geq} \iint_R m\,dA \underset{\text{Prop 3}}{=} m\iint_R 1\,dA \underset{\text{Def 3.}}{=} mA(R)$$

14.4 24) Sketch the region of integration and convert each polar integral into a Cartesian integral. Do not evaluate the integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{1}^{\csc\theta} r^2\cos\theta \, dr\, d\theta.$$

$x = r\cos\theta$, $y = r\sin\theta$.

Sol'n:



$\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

$r^2 = x^2+y^2$, so $r=1$ corresponds to the circle $x^2+y^2=1$.

$r = \csc\theta = \frac{1}{\sin\theta}$. $\Rightarrow r\sin\theta = 1$

$\Rightarrow y=1$

$\theta = \frac{\pi}{6} \Rightarrow y = \frac{x}{\sqrt{3}}$, $\theta = \frac{\pi}{2} \Rightarrow x = 0$.

$r\cos\theta = x$

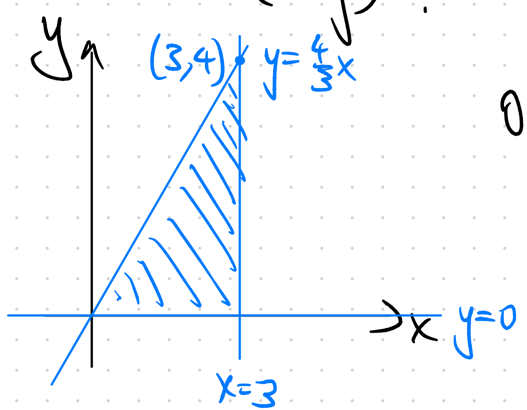
So using horizontal slices, we have $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{1}^{\csc\theta} r^2\cos\theta \, dr\, d\theta = \int_{\frac{1}{2}}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{3}y} x \, dx\, dy$.

14.4 26) Sketch the region of integration and convert each polar integral into a Cartesian integral. Do not evaluate the integral.

$$\underbrace{\int_0^{\arctan\frac{4}{3}}\int_0^{3\sec\theta} r^7\,dr\,d\theta}_{\text{I.}} + \underbrace{\int_{\arctan\frac{4}{3}}^{\frac{\pi}{2}}\int_0^{4\csc\theta} r^7\,dr\,d\theta}_{\text{II.}}.$$

<u>Sol'n:</u> $x = r\cos\theta$, $y = r\sin\theta$.

$r^7 = (x^2+y^2)^{7/2}$.



$0 \le r \le 3\sec\theta$. $r=0$

$r = 3\sec\theta = \frac{3}{\cos\theta} \Rightarrow r\cos\theta = 3$

$\Rightarrow x = 3.$

$0 \le \theta \le \arctan\frac{3}{4}$. $\theta = \arctan\frac{3}{4} \rightsquigarrow$ line $y=\frac{4}{3}x$.

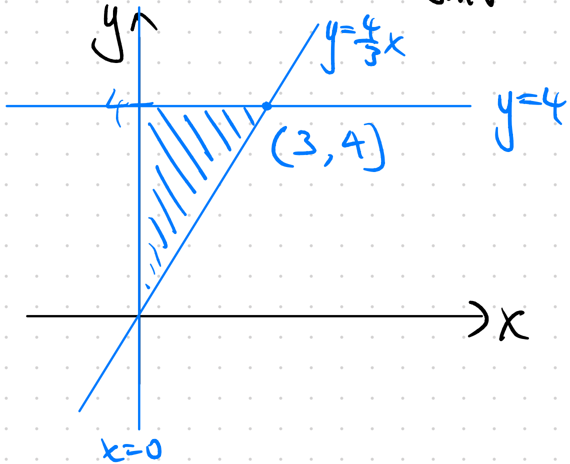
$\theta = 0 \Rightarrow$ line $y=0$.

$r^7\,dr\,d\theta = r^6\,r\,dr\,d\theta = \left((x^2+y^2)^{\frac{1}{2}}\right)^6 dy\,dx$

So $\text{I} = \int_0^{\frac{4}{3}x}\int_0^{3}(x^2+y^2)^3\,dx\,dy$

$\arctan\left(\frac{4}{3}\right) \leq \theta \leq \frac{\pi}{2} \Rightarrow$ lines $y = \frac{4}{3}x$ and $x = 0$.

$r = 4\csc\theta = \dfrac{4}{\sin\theta} \Rightarrow r\sin\theta = 4 \Rightarrow y = 4$.



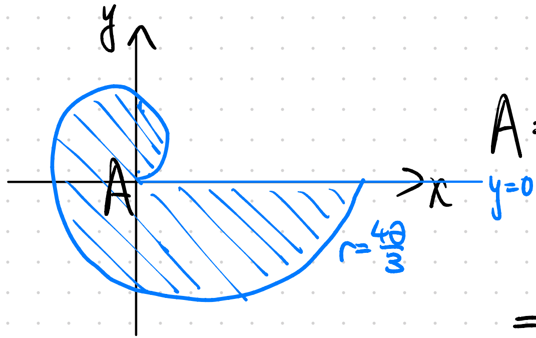
$$\text{So } I = \int_0^4 \int_0^{\frac{3}{4}y} (x^2+y^2)^3 \, dx\, dy.$$

So integral

$$= \int_0^{\frac{4}{3}x} \int_0^3 (x^2+y^2)^3 \, dx\, dy + \int_0^4 \int_0^{\frac{3}{4}y} (x^2+y^2)^3 \, dx\, dy$$

14.4 30) Find the area of the region enclosed by the positive x-axis and the spiral $r = \frac{4\theta}{3}$, $0 \le \theta \le 2\pi$. This region looks like a snail shell.

Sol'n:



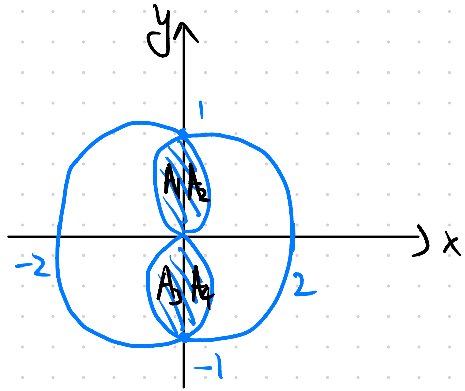
$$A = \int_0^{2\pi}\int_0^{\frac{4\theta}{3}} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=\frac{4\theta}{3}} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{16}{9}\theta^2\,d\theta = \frac{8}{9}\cdot\frac{1}{3}\theta^3\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{8}{9}\cdot\frac{1}{3}\cdot 8\pi^3 = \boxed{\frac{64\pi^3}{27}}$$

14.4 32) Find the area of the region common to the interiors of the cardioids $r = 1+\cos\theta$ and $r = 1-\cos\theta$.

Sol'n:



Note by symmetry $A = 4 \cdot A_2$, the area of the region enclosed by the cardioid $r-\cos\theta$ in the first quadrant.

When $\theta = 0$, $r = 1-1 = 0$.

$$A = 4\int_0^{\frac{\pi}{2}}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_{r=0}^{r=1-\cos\theta} d\theta = 2\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1-2\cos\theta+\cos^2\theta)\,d\theta = 2\left(\theta - 2\sin\theta\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= \pi - 4 + \left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} = \pi - 4 + \frac{\pi}{2} = \boxed{\frac{3\pi}{2} - 4}$$

14.4 34) Find the average height of the (single) cone $z=\sqrt{x^2+y^2}$ above the disk $x^2+y^2\le a^2$ in the xy-plane.

Sol'n: Area(R) $= \pi a^2$.

$x = r\cos\theta$
$y = r\sin\theta$.

Then $x^2+y^2\le a^2 \Rightarrow 0\le r^2\le a^2$
$\Rightarrow 0\le r\le a$.

The whole disk, so $0\le\theta\le 2\pi$.

$$z=\sqrt{x^2+y^2}=r.$$

So,

$$\iint_{x^2+y^2\le a^2} z\,dA = \int_0^{2\pi}\int_0^{a} r^2\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}r^3\Big|_{r=0}^{r=a}\,d\theta = \int_0^{2\pi}\frac{1}{3}a^3\,d\theta = \frac{1}{3}a^3\theta\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{2}{3}\pi a^3$$

So average height $= \dfrac{\frac{2}{3}\cancel{\pi} a^{\cancel{3}}}{\cancel{\pi}\cancel{a^2}} = \boxed{\frac{2}{3}a}$

14.4 36) Find the average value of the square of the distance from the point $P(x,y)$ in the disk $x^2+y^2 \le 1$ to the boundary point $A(1,0)$.

Sol'n: $d(x,y) = \text{dist}^2(P(x,y), A(1,0)) = \left(\left((x-1)^2+(y-0)^2\right)^{1/2}\right)^2$

$$= x^2-2x+1+y^2$$

Convert to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

so $d(r,\theta) = r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta + 1 = r^2 - 2r\cos\theta + 1$

and we integrate over the average area of the whole unit disk:

$$\text{Average dist} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 d(r,\theta)\, r\, dr\, d\theta = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 (r^3 - 2r^2\cos\theta + r)\, dr\, d\theta$$

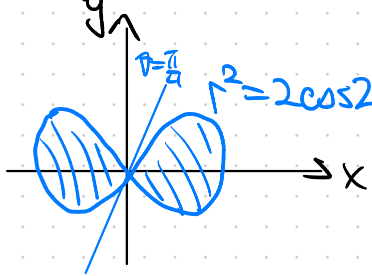
$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{4}r^4 - \frac{2}{3}r^3\cos\theta + \frac{1}{2}r^2\right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{4} - \frac{2}{3}\cos\theta + \frac{1}{2}\right) d\theta = \frac{1}{\pi}\left(\frac{1}{4}\theta - \frac{2}{3}\sin\theta + \frac{1}{2}\theta\right)\Bigg|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \pi\right) = \boxed{\frac{3}{2}}$$

14.4 40) The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2-r^2}$. Find the cylinder's volume.

Sol'n:



By symmetry, suffices to consider only the first quadrant and multiply by 4:

$$V = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2\cos 2\theta}} \sqrt{2-r^2}\cdot r\,dr\,d\theta$$

let $u = 2-r^2$
$du = -2r\,dr$
when $r=0$, $u=2$
$r = \sqrt{2\cos 2\theta}$, $u = 2-2\cos 2\theta$

$$= 4\int_0^{\frac{\pi}{4}}\int_2^{2-2\cos(2\theta)} -\frac{1}{2}u^{\frac{1}{2}}\,du\,d\theta$$

$$= -2\int_0^{\frac{\pi}{4}} \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=2}^{u=2-2\cos 2\theta}\,d\theta$$

$$= -\frac{4}{3}\int_0^{\frac{\pi}{4}}\left(\left(2-2\cos(2\theta)\right)^{3/2} - 2\sqrt{2}\right)d\theta$$

$$= -\frac{4}{3}\int_0^{\frac{\pi}{4}}\left(\left(4\sin^2\theta\right)^{3/2} - 2\sqrt{2}\right)d\theta = -\frac{4}{3}\int_0^{\frac{\pi}{4}}\left(8\sin^3\theta - 2\sqrt{2}\right)d\theta$$

$$= -\frac{4}{3}\left(8\cdot\left(\frac{1}{3}\cos^3\theta - \cos\theta\right) - 2\sqrt{2}\,\theta\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{4}}$$

$$= -\frac{4}{3}\left(8\left(\frac{1}{3}\cdot\frac{\sqrt{2}}{4} - \frac{1}{3} - \frac{\sqrt{2}}{2} + 1\right) - \frac{\sqrt{2}\pi}{2}\right)$$

$$= \boxed{\frac{2}{9}\left(20\sqrt{2} + 3\sqrt{2}\pi - 32\right)}$$

14.4 42) Evaluate the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$$

Sol'n: $r^2 = x^2+y^2$, $x \geq 0, y \geq 0$, so 1^{st} quadrant

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{1}{(1+r^2)^2}\, r\,d\theta\,dr$$

$$= \int_0^\infty \frac{r}{(1+r^2)^2}\,\theta\Big|_0^{\frac{\pi}{2}}\,dr = \frac{\pi}{2}\int_0^\infty \frac{r}{(1+r^2)^2}\,dr = \frac{\pi}{4}\int_1^\infty \frac{1}{u^2}\,du$$

let $u = 1+r^2$
$du = 2r\,dr$.

$$= \lim_{s\to\infty} \frac{\pi}{4}\int_1^s \frac{1}{u^2}\,du = \lim_{s\to\infty}\frac{\pi}{4}\left(-\frac{1}{u}\right)\Big|_1^s = \lim_{s\to\infty}\frac{\pi}{4} - \frac{\pi}{4s}$$

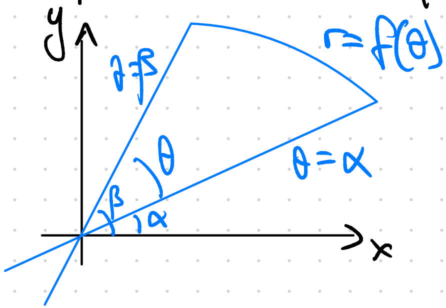
$$= \boxed{\frac{\pi}{4}}$$

14.4 44) Use the double integral in polar coordinates to derive the formula

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

for the area of the fan-shaped region between the origin and the polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$.

<u>Sol'n:</u>



$$\text{Area} = \int_{\alpha}^{\beta} \int_{0}^{f(\theta)} r \, dr \, d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \Big|_{r=0}^{r=f(\theta)} d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} f^2(\theta) \, d\theta. = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

since $r = f(\theta)$.

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